CALCULATION OF NON LIFTING POTENTIAL FLOW USING DESINGULARIZED CAUCHY'S FORMULA

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Abstract

This paper discusses the disturbance velocity and potential as well as the total velocity formulation for non lifting potential flow problem. The problem is derived based on the Cauchy method formulation. The adding and subtracting back technique is used to desingularize the integral equations. The desingularized boundary integral equations are then discretized. The discretized equations can be evaluated using an arbitrary high order Gaussian quadrature or any other numerical integration method. Numerical examples for ellipse and Joukowski airfoil are calculated and compared with the analytical solution. It is shown that the usage of a desingularized Cauchy's formula in potential flow computation leads to an efficient and an accurate determination of the velocity and pressure on the body surface.

Keywords: Boundary Element Method, Potential Flow, Cauchy Formula, Desingularization

1. Introduction

One of the effective tools in analysing potential flow about arbitrary bodies is the panel method originally presented by Hess and Smith [1]. The method utilizes a distribution of source density on the body surface and solves for the distribution necessary to meet the specific boundary conditions. Once the source density distribution is known, the velocities may be calculated. Since the panels are distributed on the boundary of the body, and in general, the boundary geometry does not have a global mathematical expression, the method is also called boundary element method (BEM). The advantage of using the BEM arises because there is no need to define a grid throughout the flow field and thus, it reduces the dimensionality of the boundary value problem.

Landweber and Macagno [2] developed a method, which differs from that of Hess and Smith mainly in the treatment of the singularity of the kernel and the procedure for obtaining numerical solution. Plane panels and quadratic panels are widely used to describe the geometry of the body to solve the boundary value problem [1, 3, 4]. Furthermore, constant source density distribution is assumed over each panel. To improve the accuracy of approximation linear and quadratic functions are also developed [1].

An arbitrary high order panel method which applies the Gaussian quadrature for descretization of the integral equation over the body has been proposed by Kouh and Ho [5]. In their method, the Gaussian quadrature points are calculated exactly from the mathematical definition of the body surface. The singularity involved in the equations and the approximation of the boundary geometry causes error in the numerical solution of the boundary element method. Regularization of the kernel for ease of computation has been studied in the past [4, 6, 7]. However, high accuracy of the numerical solution can be achieved using a proper Gaussian quadrature. By finding the limiting value of the kernel, the boundary integral can be carried out accurately. In addition,

[DOR: 20.1001.1.17357608.1386.4.6.6.7]

Hwang and Huang [8] presented a non-singular direct formulation of a boundary integral equation for potential flow using Green's identity. In their method, the singular behavior of the kernel is removed through subtracting and adding back technique from the Gauss flux theorem.

Cauchy's formula is widely used in the literature [9, 10] to solve potential problems in the field of engineering. These applications were mostly based on traditional boundary element method and local shape function. Among them, Chuang [11] developed a desingularized procedure for Cauchy's formula where a global Gaussian quadrature technique was applied to solve interior potential problem.

In this paper, the desingularized Cauchy's formula for exterior infinite domain is derived and discretized with global Gaussian quadrature. Moreover, disturbance and total complex potential and disturbance velocity of a uniform flow past a two dimensional body are formulated. The results of pressure coefficient obtained from the present numerical scheme for an ellipse and a Joukowski airfoil with different geometry are compared with the proposed analytical solution.

2. Formulation

Consider a two-dimensional body in an infinite domain R, subject to a uniform flow U_0 as shown in Fig. (1).

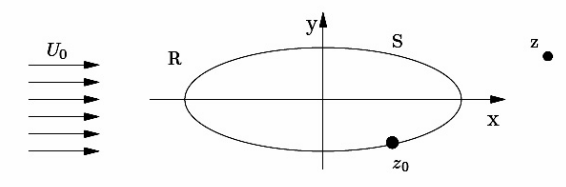

Fig. 1- Domain R and boundary S

An analytic function $\chi(z)$ in the domain R in terms of its boundary value on the boundary curve S can be expressed using Cauchy's formula. Thus, for the exterior domain R, the Cauchy's formula is given by

$$2\pi \mathrm{i}\left[\frac{k}{2\pi}-1\right]\chi(z)=\int_S \frac{\chi(z_0)}{z_0-z}dz_0 \qquad (1)$$

where

$$k=\begin{cases}0 & \text{for} \quad z\in R\\ \alpha & \text{for} \quad z\in S\end{cases}$$

and α is the angle between two tangents to the contour R at the point z. Based on the `subtracting and adding back technique' [12], the Cauchy's formula can be written in the following form

$$2\pi \mathrm{i}\left[\frac{k}{2\pi}-1\right]\chi(z)=\int_S \frac{\chi(z_0)-\chi(z)}{z_0-z}dz_0$$
$$+\chi(z)\int_S \frac{1}{z_0-z}dz_0 \qquad (2)$$

Now, using

$$\int_S \frac{1}{z_0-z}dz_0=k\,\mathrm{i} \qquad for \quad z_0\in S \qquad (3)$$

Eq. (2) reduces to Eq. (4) which is valid at any point inside the domain R, on the smooth boundary S and even at the corner point on the boundary:

$$-2\pi \mathrm{i}\,\chi(z)=\int_S \left\{\frac{\chi(z_0)-\chi(z)}{z_0-z}\right\}dz_0 \qquad (4)$$

Eq. (4) has a finite value when z_0 approaches z on the boundary S. Hence,

$$\lim_{z_0\to z}\frac{\chi(z_0)-\chi(z)}{z_0-z}dz_0=\left.\frac{d\chi(z_0)}{dz_0}\right|_{z_0=z}$$
$$=\chi'(z) \qquad (5)$$

The integral in Eq. (4) is a non singular integral and can be carried out using an arbitrary Gaussian quadrature with N Gaussian points:

$$\sum_{j=1}^{N}\frac{\chi_j-\chi(z)}{z_j-z}dz_j w_j=-2\pi\mathbf{i}\chi(z)$$
$$for \quad z\neq z_j \qquad (6)$$

where χ_j and w_j are the function value of $\chi(z)$ and weighting factor, respectively, at Gaussian point z_j.

Eq. (6) is applicable on the boundary and inside the domain. Moreover, the value of function $\chi(z)$ at the Gaussian points on the boundary can be computed accurately using Eq. (6) as discussed in the following sections. If the value of the function $\chi(z)$ is known on the boundary, then the function may be easily and accurately computed at any point inside the domain.

2.1 Discretization Using Gaussian Quadrature

To solve the unknown part of $\chi(z)$ on the boundary, the field point z should be located on the boundary S. Then, integral Eq. (4) can be expressed in terms of the arc length of S as

$$-2\pi\,\mathbf{i}\,\chi(z(s)) = \int_S \left\{ \frac{\chi(z_0(s)) - \chi(z(s))}{z_0 - z} \right\} \frac{dz_0}{ds}\, ds \qquad for \quad z,\ z_0 \in S \qquad (7)$$

In Eq. (7), the limiting value of the integrand when z_0 approaches z on the boundary is equal to

$$\lim_{z_0 \to z} \left\{ \frac{\chi(z_0(s)) - \chi(z(s))}{z_0 - z} \right\} \frac{dz_0}{ds} = \frac{d\chi}{dz_0}\frac{dz_0}{ds} = \frac{d\chi}{ds} \qquad (8)$$

Since the limiting value of the integrand in Eq. (4) exists, this equation is a non-singular representation of the boundary integral equation. Applying Gaussian quadrature with N Gaussian points which are also selected as the collocation points, to the integral Eq. (7), a system of complex equations can be obtained:

$$\sum_{j=1, j\neq i}^{N} \left\{ \frac{\chi_j - \chi_i}{z_j - z_i} \right\} \left. \frac{dz_0}{ds} \right|_j w_j + \left. \frac{d\chi}{ds} \right|_i w_i = -2\pi\,\mathbf{i}\,\chi_i, \qquad i = 1,\ldots,N \qquad (9)$$

2.1.1 Formulation of Disturbance and Total Velocity Potential

If the disturbance complex potential of a uniform flow past a body is given by $\chi(z)$ in the z-plane, then

$$\chi(z(s)) = \phi(s) + \mathbf{i}\,\psi(s)$$
$$z(s) = x(s) + \mathbf{i}\,y(s)$$

And

$$\frac{d\chi}{ds} = \frac{d\phi}{ds} + \mathbf{i}\frac{d\psi}{ds} = \phi' + \mathbf{i}\,\psi'$$
$$\frac{dz}{ds} = \frac{dx}{ds} + \mathbf{i}\frac{dy}{ds} = x' + \mathbf{i}\,y' \qquad (10)$$

where the real and imaginary parts of dz/ds represent the components of the tangent vector $\vec{s}$ along the boundary curve S.

Substituting Eq. (10) into Eq. (9) gives

$$\sum_{j=1, j\neq i}^{N} \left\{ \frac{(\phi_j - \phi_i) + \mathbf{i}(\psi_j - \psi_i)}{(x_j - x_i) + \mathbf{i}(y_j - y_i)} \right\} (x_j' - \mathbf{i}\,y_j')\,w_j + (\phi_{ij}' + \mathbf{i}\,\psi_i')\,w = -2\pi\mathbf{i}\,(\phi_i + \mathbf{i}\,\psi_i) \qquad for \quad i = 1,\ldots,N \qquad (11)$$

After some manipulation on Eq. (11), Eqs. (12) and (13) are obtained in the following matrix form

$$\sum_{j=1}^{N} (A_{ij}\psi_j) = \sum_{j=1}^{N} (B_{ij}\varphi_j) - w_i\phi_i' \qquad \textbf{for} \quad i = 1,\ldots,N \qquad (12)$$

$$\sum_{j=1}^{N} (A_{ij}\phi_j) = -\sum_{j=1}^{N} (B_{ij}\psi_j) - w_i\psi_i' \qquad \textbf{for} \quad i = 1,\ldots,N \qquad (13)$$

where

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$$\begin{cases} A_{ij} = \dfrac{\vec{r}_{ij}.\vec{n}_j}{r_{ij}^2} w_j \\ A_{ii} = 2\pi - \displaystyle\sum_{j=1, j\neq i}^{N} \dfrac{\vec{r}_{ij}.\vec{n}_j}{r_{ij}^2} w_j \end{cases}$$

$$\begin{cases} B_{ij} = \dfrac{\vec{r}_{ij}.\vec{n}_j}{r_{ij}^2} w_j \\ B_{ij} = -\displaystyle\sum_{j=1, j\neq i}^{N} \dfrac{\vec{r}_{ij}.\vec{n}_j}{r_{ij}^2} w_j \end{cases}$$

$\vec{r}_{ij} = (x_j - x_i)\vec{i} + (y_j - y_i)\vec{j}$, $\vec{n}_j$ and $\vec{s}_j$ are the unit normal and tangential vectors at point j respectively.

To compute the total velocity potential, the exterior incident flow potential ϕ_i can be extended into the interior region; then, the interior region becomes the domain [8]. Using the Cauchy's formula for interior domain [4] and adding the incident potential to disturbance potential Eqs. (12) and (13) reduce to a versatile form for total complex potential as shown in:

$$\sum_{j=1}^{N} (B_{ij}\Phi_j) = -w\Phi_i' - 2\pi\Psi_{Ii}, \qquad (14)$$

$$\sum_{j=1}^{N} (A_{ij}\Phi_j) = 2\pi\Phi_{Ii}, \qquad (15)$$

where Φ_i is the total potential at the i th node , ϕ_{Ii} and ψ_{Ii} are the incident velocity potential and stream function at the i th node respectively.

2.1.2 Formulation of Complex velocity

If the complex disturbance velocity of a uniform flow past a body is given by $w(z) = u(x,y) - \mathbf{i}v(x,y)$ in the z-plane, the desingularized Cauchy integral equation for complex velocity can be written as

$$-2\pi\mathbf{i}w(z) = \int_S \frac{w(z) - w(z)}{z - z} \frac{dz}{ds}\bigg|_j w_j dz_0,$$

$$z,\ z_0 \in S \qquad (16)$$

discretizing Eq. (16) with Gaussian quadrature, one obtains

$$-2\pi\mathbf{i}w(z) = \sum_{j=1, j\neq i}^{N} \frac{w(z) - w(z)}{z - z} \frac{dz}{ds}\bigg|_j w_j + \frac{dw}{ds}\bigg|_i \frac{dz}{ds}\bigg|_i w_i \qquad (17)$$

Additionally, the complex velocity $w(z)$ may be expressed as

$$w(z) = \frac{d\chi}{dz} = \frac{\phi' + \mathbf{i}\psi'}{x' - \mathbf{i}y'} = u(x,y) - \mathbf{i}v(x,y) \qquad (18)$$

and

$$\frac{dw}{ds} = \frac{du}{ds} - \mathbf{i}\frac{dv}{ds} = u' - \mathbf{i}v' \qquad (19)$$

$u(s)$ and $v(s)$ are the real and imaginary parts of the velocity along the boundary S, respectively, and can be written

$$u(s) = \phi' x' + \psi' y'$$
$$v(s) = \phi' y' - \psi' x' \qquad (20)$$

Using Eq. (20), the kinematic boundary condition along the boundary S might be specified as

$$\vec{V}(s).\vec{n}(s) = \left[u(s)\vec{i} + v(s)\vec{j}\right]\left[y'(s)\vec{i} + x'(s)\vec{j}\right] = -\frac{dy}{ds} \qquad (21)$$

Thus, the relation between velocity components along the boundary is obtained as:

$$v(s) = \frac{y'}{x'}(u(s) + 1) \qquad (22)$$

Substituting Eqs. (18) and (19) into Eq. (17) gives

$$\sum_{j=1, j\neq i}^{N}\left\{\frac{(u_j-u_i)-\mathbf{i}(v_j-v_i)}{(x_j-x_i)+\mathbf{i}(y_j-y_i)}\right\}(x_j^{'}+\mathbf{i}y_j^{'})w_j$$
$$+(u_i^{'}-\mathbf{i}v_i^{'})w=-2\pi\mathbf{i}(u_i-\mathbf{i}v_i), \qquad (24)$$

Separating the real part and imaginary part of Eq. (24) two sets of real equations are obtained

$$-\sum_{j=1}^{N}(A_{ij}v_j)=\sum_{j=1}^{N}(B_{ij}u_j)+w_iu_i^{'}, \qquad (25)$$

$$\sum_{j=1}^{N}(A_{ij}u_j)=\sum_{j=1}^{N}(B_{ij}v_j)+w_iv_i^{'}, \qquad (26)$$

Substituting v and $v^{'}$ from Eqs. (22) and (23) into Eqs. (25) and (26), $2N$ equations with $2N$ unknowns, namely, u_j and $u_j^{'}$ are obtained

$$[C_{ij}]\{u_j\}=D_i+w_iu_i^{'} \qquad i=1,...,N \qquad (27)$$

$$[E_{ij}]\{u_j\}=F_i+G_iw_iu_i^{'} \qquad i=1,...,N \qquad (28)$$

where

$$C_{ij}=-A_{ij}\frac{y_j^{'}}{x_j^{'}}-B_{ij} \quad and \quad D_i=A_{ij}\frac{y_j^{'}}{x_j^{'}}$$

$$E_{ij}=\begin{cases}A_{ij}-B_{ij}\dfrac{y_j^{'}}{x_j^{'}}\\ A_{ij}-B_{ij}\dfrac{y_j^{'}}{x_j^{'}}-w_i\left[\dfrac{x_i^{'}y_i^{''}-x_i^{''}y_i^{'}}{x_i^{'2}}\right]\end{cases}$$

$$F_{ij}=B_{ij}\frac{y_j^{'}}{x_j^{'}}+w_i\left[\frac{x_i^{'}y_i^{''}-x_i^{''}y_i^{'}}{x_i^{'2}}\right] \qquad G_i=\frac{y_i^{'}}{x_i^{'}}$$

A_{ij} and B_{ij} are the same as in Eqs. (12) and (13).

3. Numerical Examples

The discretized form of the desingularized Cauchy formula is solved and simulated for two kind of body geometry. One is an elliptical cylinder and the other a Joukowski airfoil. The discussion of simulation results are in the following sections.

3.1. Uniform Flow past an Elliptical Cylinder

The boundary of the elliptical cylinder is computed from a conformal mapping given by

$$\varsigma(\theta)=e^{i\theta}+\frac{c}{e^{i\theta}} \qquad 0\leq\theta\leq 2\pi \qquad (29)$$

and the complex potential $\Omega(\varsigma(\theta))$ on the boundary of the elliptical cylinder is given by

$$\Omega(\varsigma(\theta))=U\left\{\varsigma(\theta)+\frac{1}{\varsigma(\theta)}\right\} \qquad (30)$$

In the computational procedure, the Gaussian points are used as the collocation points, and the corresponding parameter θ of the collocation points are computed using the Newton Raphson iteration technique.

$$\theta_j^{(n+1)}=\theta_j^{(n)}-\frac{\Omega(\theta_j^{n})}{\Omega^{'}(\theta_j^{(n)})} \qquad (31)$$

where n and $(n+1)$ denotes the nth and the $(n+1)$th iterations respectively.

Now, the boundary conditions which are $\psi_j=-y_j$ and $\psi_j^{'}=-[dy/ds]_j$ will be imposed at the collocation points. Finally, the complex disturbance velocity, w_j, at collocation point s_j, for $j=1,...,N$:

$$w_j=u_j-\mathbf{i}v_j=\left.\frac{d\Omega}{dz}\right|_j=\frac{\phi_j^{'}+\mathbf{i}\psi_j^{'}}{x_j^{'}+\mathbf{i}y_j^{'}} \qquad (32)$$

and the pressure coefficient $C_P^{(j)}$ at collocation point s_j, for $j=1,...,N$ is:

$$C_P^{(j)}=1-\frac{[(U+u_j)x_j^{'}+v_jy_j^{'}]^2}{U^2} \qquad (33)$$

where U is the velocity of the uniform flow. Based on this procedure, three different elliptical cylinders are chosen, corresponding to $c=0.3, 0.5$ and 0.7.

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Each elliptical cylinder has the major semi-axis and the minor semi-axis of $1+c$ and $1-c$ respectively. Moreover, the boundary of each elliptical cylinder is divided into four equal parts on which 4, 8, 12, 16 and 30-points Gaussian quadratures are applied. In addition, the root mean square errors for pressure coefficient, RMS, are calculated as follows;

$$RMS = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left\{\frac{C_{p_{exact}}^{(j)} - C_{p_{num}}^{(j)}}{C_{p_{exact}}^{(j)}}\right\}^2} \qquad (34)$$

where, the pressure coefficient, $C_{p_{exacy}}$, at the Gaussian points is computed based on the above procedure and $C_{p_{num}}$ is computed using proposed complex potential formulation and complex velocity formulation. N is the number of Gaussian points in the entire length of the body surface.

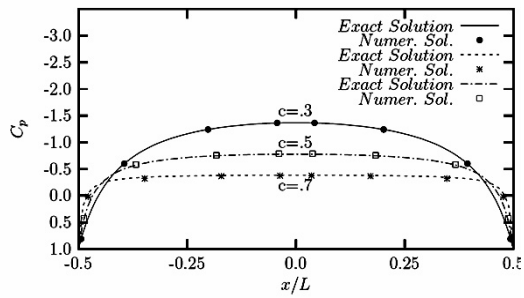

Fig. 2- Pressure coefficient vs. non-dimensional length, x/L, for ellipse with different c.

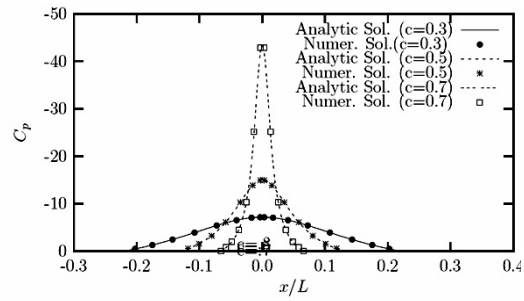

Fig. 3- Pressure coefficient for a vertical ellipse with different c.

Fig. (2) shows the distribution of pressure coefficient on the surface of an ellipse with varying c. It indicates that even with a few number of Gaussian points the results obtained from the proposed methods are very accurate. Root mean square error of pressure coefficient is depicted in Fig. (4). It is indicated that the velocity formulation gives more accurate results than the potential formulation. However, these results show that increasing the number of collocation points beyond a certain threshold does not significantly improve the accuracy of computation.

Fig. (4) also shows that the RMS error of pressure coefficients increase by increasing the value of c for an elliptical cylinder.

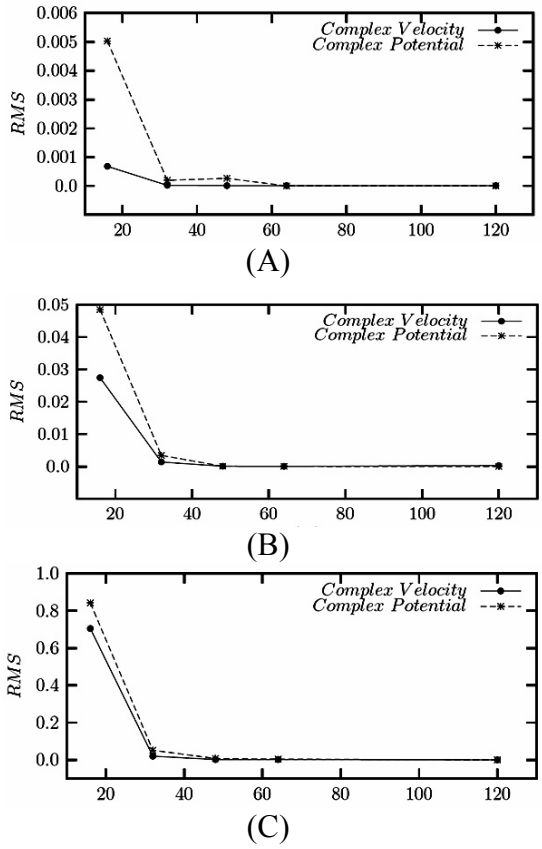

Fig. 4- RMS of pressure coefficient, C_p, for an ellipse; (A) c=0.3, (B) c=0.5, (C) c=0.7

3.2 Uniform Flow past a Joukowski Airfoil

The mapping function which maps a unit circle to a Joukowski airfoil is given by

$$\varsigma(\theta) = e^{i\theta} + \sqrt{c} - e^{i\beta} + \frac{c}{e^{i\theta} + \sqrt{c} - e^{i\beta}} \quad (35)$$

where $\beta = 0$ for nonlifting case.

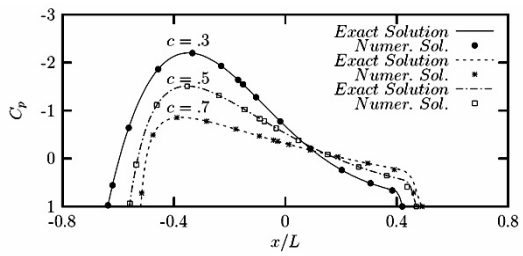

Fig. 5- pressure coefficient, C_p ,for Joukowski airfoil with different c.

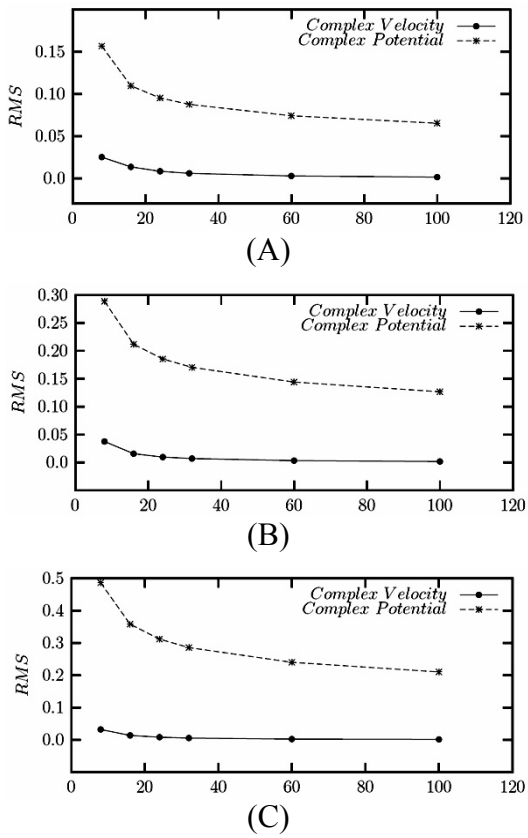

Fig. 4- RMS of pressure coefficient, C_p, for Joukowski airfoil; (A) c=0.3, (B) c=0.5, (C) c=0.7

The complex potential on the boundary of the airfoil and the computational procedure in this example is the same as that of the elliptical cylinder in the previous section.

In the numerical simulation, three different Joukowski airfoils are computed, corresponding to $c = 0.3, 0.5$ and 0.7. The total arc length of the airfoil is divided into two equal parts on which 4, 8, 12, 16, 30 and 50-points Gaussian quadratures are again applied. The pressure coefficient for a Joukowski airfoil is plotted in Fig. (5). It is shown that the numerical solutions are in good agreement with the analytic solution. However, since the thickness at the trailing edge of the Joukowski airfoil is zero, all BEM codes encounter numerical difficulties in the computation. Fig. (6) shows the RMS error of pressure coefficient for Joukowski airfoil. It indicates that the velocity formulation gives more accurate results than complex potential formulation.

It also indicates that increasing the number of collocation points beyond a certain threshold does not significantly improve the accuracy of computation.

4. Conclusions

Desingularized Cauchy's formula for computation of two dimensional potential flows in an infinite domain is presented. In addition, based on the desingularized Cauchy's formula and Gaussian quadrature, the numerical schemes are developed for solving disturbance and total velocity potential as well as the complex velocity.

Numerical simulations show that the accuracy of the numerical solution obtained from the proposed numerical schemes are sensitive to the number of collocation points used in the Gaussian quadrature, but in general, the sensitivity can be reduced by increasing the number

[DOR: 20.1001.1.17357608.1386.4.6.6.7]

of collocation points. Since the normal and tangential vectors and their derivatives at the collocation points are involved in the numerical solution, the boundary geometry is important in the numerical solution. While the equation of the total velocity potential is simpler than those of the velocity potential and the disturbance velocity, the use of the total velocity potential formulation reduces the computational procedure.

In addition, numerical simulations indicate that the numerical solutions for the pressure coefficient obtained from the complex disturbance velocity formulation are usually more accurate than those obtained from the complex disturbance or total potential. Numerical results for the disturbance and total velocity potential are usually of the same order. Since the equations of the total velocity potential are simpler than those of the disturbance velocity potential and the disturbance velocity, the use of the total velocity potential formulation reduces the computational procedure.

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